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ON THE APPROXIMATION OF LOGARITHMS OF ALGEBRAIC NUMBERS

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A new identity is given by means of which infinitely many algebraic functions approximating the logarithmic function $\ln x$ are obtained. On substituting numerical algebraic values for the variable, a lower bound for the distance of its logarithm from variable algebraic numbers is found. As a further application, it is proved that the fractional part of the number e^a is greater than a^{-40a} for every sufficiently large positive integer a.

After earlier and weaker results by Mordukhay-Boltowskoy (1923), Siegel (1929) and Popken (1929), I proved in 1931 (Mahler 1932) that $\ln x$, for rational $x \neq 0, \neq 1$, is not a Liouville number and even not a U-number, and I determined a measure of transcendency for such logarithms. Up to now this measure has not been improved, although Fel'dman (1951) recently proved a very general related inequality for the logarithms of *arbitrary* algebraic numbers.

In this paper, I once more study the question of approximations to $\ln x$. The new work is based on a simple system of identities I found a year ago. These are of the form

$$\sum_{k=0}^{m} A_{hk}(x)\,(\ln x)^k = R_h(x) \quad (h = 0, 1, \ldots, m), \tag{A}$$

where the A's are polynomials of degree not greater than n with integral coefficients and of determinant

$$c(x-1)^{(m+1)n} \quad (c \neq 0),$$

while the R's have at $x = 1$ a zero of order at least $(m+1)\,n$. From the integral defining $R_h(x)$ one easily derives upper bounds for $|A_{hk}(x)|$ and $|R_h(x)|$.

Let now $\xi \neq 0, \neq 1$, be an algebraic number which need not be a constant. On allowing m and n to vary, the identities (A) become infinitely many approximative algebraic equations for $\ln \xi$ with algebraic coefficients. By means of these, it is proved in Chapter 2 that $\ln \xi$ *is not a U-number*. In this way my old result has for the first time been extended to *arbitrary* algebraic numbers. Moreover, the new proof is much simpler than the old one. It is based on an idea due to Siegel (1929). It may be mentioned that the measure of transcendency now obtained does not contain any unknown numerical constants.

In Chapter 3 this measure is further improved under the restrictive assumption that both ξ and the approximations to $\ln \xi$ are rational numbers. As an application, it is proved that

$$|2^a - e^{a_1}| \geq |2^3 - e^2|$$

for all pairs of positive integers a and a_1.

In the last chapter, I finally apply the identities (A) to prove that

$$e^a - [e^a] > a^{-40a}, \quad \ln f - [\ln f] > f^{-40 \ln \ln f},$$

when both a and f are sufficiently large positive integers; here $[x]$ denotes the integral part of x.

The formulae (A) may also be used to show that

$$|a_0+a_1 e+\ldots+a_m e^m|>e^{-cm},$$

when m is a positive integer tending to infinity, $a_0, a_1, \ldots, a_m$ are m bounded integers not all zero, and $c>1$ does not depend on the a's or on m. However, this result is very weak and it would therefore be of great interest to replace it by a better one.

Chapter 1. The approximation functions

1. Let m and n be two positive integers and $x\neq 0$ and z two complex variables. We define x^z by

$$x^z = e^{z \ln x},$$

where $\ln x$ stands for the principal value of the logarithm which is real when x is a positive number. We further denote by

$$N=[1,2,\ldots,n]$$

the least common multiple of $1, 2, \ldots, n$, and put, for shortness,

$$P=m!\,N^m(n!)^{m+1}.$$

Finally, let $Q(z)$ be the polynomial

$$Q(z)=\{z(z+1)\ldots(z+n)\}^{m+1}.$$

We study in this chapter the integrals

$$R_h(x)=\frac{P}{2\pi i}\int_C \frac{z^h x^{z+n}}{Q(z)}\,dz \quad (h=0,1,\ldots,m),$$

extended over the circle C in the complex z-plane of centre $z=0$ and arbitrary radius ρ greater than n, described in the positive direction. In the next two sections, $R_h(x)$ will be evaluated in two different ways. The resulting identity will lead us to the approximation formulae needed in the following chapters.

2. The rational function $Q(z)^{-1}$ has at $z=\infty$ a zero of order $(m+1)(n+1)$, and its poles are at points of absolute value not greater than n. The function possesses therefore a Laurent expansion

$$Q(z)^{-1}=\sum_{\kappa=(m+1)(n+1)}^{\infty} c_\kappa z^{-\kappa},$$

convergent for $|z|>n$. The other factor, $z^h x^{z+n}$, of the integrand can be developed into the power series

$$z^h x^{z+n}=x^n\sum_{\kappa=0}^{\infty}\frac{(\ln x)^\kappa}{\kappa!}z^{\kappa+h},$$

which converges for all z. Hence, on multiplying these two series and integrating the product series term by term, we obtain for $R_h(x)$ the convergent development

$$R_h(x)=Px^n\sum_{\kappa=(m+1)(n+1)}^{\infty} c_\kappa\frac{(\ln x)^{\kappa-h-1}}{(\kappa-h-1)!} \quad (h=0,1,\ldots,m).$$

It shows that $R_h(x)$ vanishes at $x=1$ to the exact order

$$(m+1)(n+1)-h-1\geqslant(m+1)\,n,$$

because $\ln x$ has at this point a zero of the first order.

3. By the residue theorem, $R_h(x)$ may also be written as

$$R_h(x) = P\sum_{\lambda=0}^{n} r_\lambda,$$

where r_λ denotes the residue of the integrand

$$\frac{z^h x^{z+n}}{Q(z)}$$

at the pole $z=-\lambda$. This residue is evaluated as follows.

At $z=-\lambda$, $Q(z)$ has a zero of order $m+1$. Hence $S_h(z) = \dfrac{z^h}{Q(z)}$ has at $z=-\lambda$ a pole of order not greater than $m+1$, and the other poles of this function lie at least at a distance 1 from this point. Therefore $S_h(z)$ can be developed into a Laurent series

$$S_h(z) = \sum_{\kappa=-m-1}^{\infty} \gamma_\kappa^{(\lambda,h)}(z+\lambda)^\kappa$$

convergent inside the circle of centre $z=-\lambda$ and radius 1. On the other hand,

$$x^{z+n} = x^{n-\lambda}\sum_{\kappa=0}^{\infty}\frac{(\ln x)^\kappa}{\kappa!}(z+\lambda)^\kappa$$

for all values of z. Hence, on multiplying these two series term by term, the residue r_λ is found to be equal to

$$r_\lambda = x^{n-\lambda}\sum_{\kappa=0}^{m}\gamma_{-\kappa-1}^{(\lambda,h)}\frac{(\ln x)^\kappa}{\kappa!},$$

whence
$$R_h(x) = P\sum_{\kappa=0}^{m}\sum_{\lambda=0}^{n}\gamma_{-\kappa-1}^{(\lambda,h)}x^{n-\lambda}\frac{(\ln x)^\kappa}{\kappa!} \quad (h=0,1,\ldots,m).$$

We therefore put
$$A_{hk}(x) = \frac{P}{k!}\sum_{\lambda=0}^{n}\gamma_{-\kappa-1}^{(\lambda,h)}x^{n-\lambda} \quad (h,k=0,1,\ldots,m),$$

and have
$$R_h(x) = \sum_{k=0}^{m}A_{hk}(x)(\ln x)^k \quad (h=0,1,\ldots,m)$$
identically in x.

4. The functions $A_{hk}(x)$ are polynomials in x, their terms of highest degree being

$$\frac{P}{k!}\gamma_{-k-1}^{(0,h)}x^n.$$

This term can be obtained more explicitly as follows.

Write the Laurent series for $S_0(z)=Q(z)^{-1}$ at $z=0$ in the simpler form

$$Q(z)^{-1} = \sum_{\kappa=-m-1}^{\infty}\gamma_\kappa z^\kappa.$$

Then, from $Q(z) = \{z(z+1)\ldots(z+n)\}^{m+1}$,

$$\gamma_{-m-1} = (n!)^{-(m+1)},$$

and the coefficients $\gamma_\kappa^{(0,h)}$ of the more general function $S_h(z)=z^hS_0(z)$ are given by

$$\gamma_\kappa^{(0,h)} = \begin{cases} 0 & \text{if } -m-1\leqslant\kappa\leqslant -m+h-2, \\ \gamma_{\kappa-h} & \text{if } \kappa\geqslant -m+h-1. \end{cases}$$

It is therefore obvious that $A_{hk}(x)$ is of smaller degree than n if $h+k \geqslant m+1$; that it is of exact degree n and has the highest term

$$\frac{P}{k!}(n!)^{-(m+1)}x^n$$

if $h+k=m$; and that its degree does not exceed n if $h+k \leqslant m-1$.

5. The last remarks enable us to evaluate the determinant

$$D(x)=\begin{vmatrix} A_{00}(x) & A_{01}(x) & \dots & A_{0m}(x) \\ A_{10}(x) & A_{11}(x) & \dots & A_{1m}(x) \\ \vdots & \vdots & & \vdots \\ A_{m0}(x) & A_{m1}(x) & \dots & A_{mm}(x) \end{vmatrix}.$$

If, to start with, the elements of $D(x)$ are replaced by their highest terms

$$\frac{P}{k!}\gamma_{-k-1}^{(0,h)}x^n,$$

and the terms of lower degree omitted, we obtain a triangular determinant with elements 0 below the diagonal $h+k=m$, hence equal to

$$\mp\prod_{k=0}^{m}\left\{\frac{P}{k!}\gamma_{-k-1}^{(0,m-k)}x^n\right\}=\mp\frac{P^{m+1}(n!)^{-(m+1)^2}}{1!\,2!\dots m!}x^{(m+1)n}.$$

Therefore $D(x)$ itself is of the form

$$D(x)=\mp\frac{P^{m+1}(n!)^{-(m+1)^2}}{1!\,2!\dots m!}x^{(m+1)n}+\text{terms in lower powers of } x.$$

To obtain these lower terms, add to the first column of $D(x)$ the second column times $\ln x$, the third column times $(\ln x)^2$, etc., finally, the last column times $(\ln x)^m$. By the identities

$$\sum_{k=0}^{m}A_{hk}(x)\,(\ln x)^k=R_h(x)\quad (h=0,1,\dots,m),$$

the new first column consists then of the elements

$$R_0(x),\quad R_1(x),\quad \dots,\quad R_m(x),$$

all of which, by [2], vanish at $x=1$ to at least the order $(m+1)\,n$. Since the other elements of $D(x)$ are regular at $x=1$, the determinant is then necessarily divisible by $(x-1)^{(m+1)n}$. By the form of its highest term, $D(x)$ must then be identical with

$$D(x)=\mp\frac{P^{m+1}(n!)^{-(m+1)^2}}{1!\,2!\dots m!}(x-1)^{(m+1)n}.$$

Hence, in particular, $D(x)\neq 0$ if $x\neq 1$,

a result we shall frequently apply in later chapters.

6. We next investigate the arithmetical form of the coefficients $\gamma_{-k-1}^{(\lambda,h)}$ in

$$A_{hk}(x)=\frac{P}{k!}\sum_{\lambda=0}^{n}\gamma_{-k-1}^{(\lambda,h)}x^{n-\lambda}.$$

These coefficients were originally obtained from the Laurent expansion

$$S_h(z) = z^h Q(z)^{-1} = \sum_{\kappa=-m-1}^{\infty} \gamma_\kappa^{(\lambda,h)} (z+\lambda)^\kappa.$$

Now, for $\lambda = 0, 1, \ldots, n$, $Q(z)^{-1}$ can be written as

$$Q(z)^{-1} = (z+\lambda)^{-m-1} \prod_{\mu=1}^{\lambda} \{(z+\lambda)-\mu\}^{-m-1} \prod_{\nu=1}^{n-\lambda} \{(z+\lambda)+\nu\}^{-m-1},$$

and so also as

$$Q(z)^{-1} = (-1)^{\lambda(m+1)} \{\lambda!(n-\lambda)!\}^{-m-1} (z+\lambda)^{-m-1} \prod_{\mu=1}^{\lambda} \left(1-\frac{z+\lambda}{\mu}\right)^{-m-1} \prod_{\nu=1}^{n-\lambda} \left(1+\frac{z+\lambda}{\nu}\right)^{-m-1}.$$

Further, by the definition of N as the least common multiple of $1, 2, \ldots, n$, the quotients

$$\frac{N}{\mu} \text{ and } \frac{N}{\nu}, \quad \text{where} \quad \mu = 1, 2, \ldots, \lambda \text{ and } \nu = 1, 2, \ldots, n-\lambda,$$

are integers; there exist therefore integral coefficients $\alpha_\kappa^{(\lambda)}$ such that

$$\prod_{\mu=1}^{\lambda} \left(1-\frac{N}{\mu}t\right)^{-m-1} \prod_{\nu=1}^{n-\lambda} \left(1+\frac{N}{\nu}t\right)^{-m-1} = \sum_{\kappa=0}^{\infty} \alpha_\kappa^{(\lambda)} t^\kappa.$$

Hence, from the product for $Q(z)^{-1}$,

$$Q(z)^{-1} = (-1)^{\lambda(m+1)} \{\lambda!(n-\lambda)!\}^{-m-1} \sum_{\kappa=0}^{\infty} \alpha_\kappa^{(\lambda)} N^{-\kappa} (z+\lambda)^{\kappa-m-1}.$$

On multiplying this series by

$$z^h = \{(z+\lambda)-\lambda\}^h = \sum_{\kappa=0}^{h} \binom{h}{\kappa} (-\lambda)^{h-\kappa} (z+\lambda)^\kappa,$$

it is evident that the coefficients $\gamma_\kappa^{(\lambda,h)}$ can be written as

$$\gamma_\kappa^{(\lambda,h)} = (-1)^{\lambda(m+1)} \{\lambda!(n-\lambda)!\}^{-m-1} \sum \binom{h}{\kappa_1} (-\lambda)^{h-\kappa_1} \alpha_{\kappa_2}^{(\lambda)} N^{-\kappa_2},$$

where the summation extends over all pairs of integers κ_1, κ_2 satisfying

$$0 \leqslant \kappa_1 \leqslant h, \ \kappa_2 \geqslant 0, \ \kappa_1+\kappa_2 = \kappa+m+1, \quad \text{hence also} \quad \kappa_2 \leqslant \kappa+m+1.$$

In this formula

$$\{\lambda!(n-\lambda)!\}^{-m-1} = (n!)^{-m-1} \binom{n}{\lambda}^{m+1}$$

is a rational number the denominator of which divides $(n!)^{m+1}$. It follows therefore that

$$(n!)^{m+1} N^{\kappa+m+1} \gamma_\kappa^{(\lambda,h)}$$

is a rational integer. In particular, all products

$$(n!)^{m+1} N^{m-k} \gamma_{-k-1}^{(\lambda,h)} \quad \begin{pmatrix} h, k = 0, 1, \ldots, m \\ \lambda = 0, 1, \ldots, n \end{pmatrix}$$

are integers, hence even more all products

$$(n!)^{m+1} N^m \gamma_{-k-1}^{(\lambda,h)} \quad \begin{pmatrix} h, k = 0, 1, \ldots, m \\ \lambda = 0, 1, \ldots, n \end{pmatrix}.$$

Since, for $k = 0, 1, \ldots, m$, $k!$ is a divisor of $m!$, we obtain then the final result that *all the* $(m+1)^2$ *polynomials*

$$A_{hk}(x) = \frac{m!}{k!} N^m (n!)^{m+1} \sum_{\lambda=0}^{n} \gamma_{-k-1}^{(\lambda,h)} x^{n-\lambda} \quad (h, k = 0, 1, \ldots, m)$$

have rational integral coefficients. This property will prove of importance in the later applications.

7. If $p(x) = p_0 + p_1 x + p_2 x^2 + \ldots + p_r x^r$ is a polynomial with arbitrary real or complex coefficients, then we write

$$\overline{|p(x)|} = \max(|p_0|, |p_1|, \ldots, |p_r|),$$

and call $\overline{|p(x)|}$ the *height* of $p(x)$. Our next aim is to find an estimate for the height of $A_{hk}(x)$.

This requires obtaining an upper bound for the Laurent coefficients $\gamma_\kappa^{(\lambda,h)}$ in

$$S_h(z) = z^h Q(z)^{-1} = \sum_{\kappa=-m-1}^{\infty} \gamma_\kappa^{(\lambda,h)} (z+\lambda)^\kappa$$

when $\kappa = -1, -2, \ldots, -m-1$.

By Cauchy's theorem,
$$\gamma_\kappa^{(\lambda,h)} = \frac{1}{2\pi i}\int_{C_\lambda} \frac{z^h\,dz}{Q(z)\,(z+\lambda)^{\kappa+1}},$$

where C_λ may be chosen as the circle of centre $z = -\lambda$ and radius $\frac{1}{2}$, described in the positive direction. Since $0 \leqslant \lambda \leqslant n$ and $0 \leqslant h \leqslant m$, we have on this circle,

$$|z| \leqslant \lambda + \tfrac{1}{2} < n+1, \quad \text{hence} \quad |z^h| < (n+1)^m,$$

further
$$|(z+\lambda)^{-\kappa-1}| \leqslant 1,$$
because $\kappa + 1 = -k \leqslant 0$ in $A_{hk}(x)$.

Next,
$$Q(z) = \left\{\prod_{\mu=1}^{\lambda}\Big((z+\lambda)-\mu\Big)\right\}^{m+1} (z+\lambda)^{m+1} \left\{\prod_{\nu=1}^{n-\lambda}\Big((z+\lambda)+\nu\Big)\right\}^{m+1},$$

so that for all points on C_λ,

$$|Q(z)| \geqslant \left\{\prod_{\mu=1}^{\lambda}(\mu-\tfrac{1}{2})\right\}^{m+1} (\tfrac{1}{2})^{m+1} \left\{\prod_{\nu=1}^{n-\lambda}(\nu-\tfrac{1}{2})\right\}^{m+1} = \left\{\frac{(2\lambda)!}{\lambda!\,2^{2\lambda}}\right\}^{m+1} (\tfrac{1}{2})^{m+1} \left\{\frac{(2n-2\lambda)!}{(n-\lambda)!\,2^{2n-2\lambda}}\right\}^{m+1}$$
$$= \left\{\frac{(2\lambda)!\,(2n-2\lambda)!}{(2n)!}\,\frac{(2n)!}{n!\,n!}\,\frac{n!}{\lambda!(n-\lambda)!}\,n!\,\frac{1}{2^{2\lambda+(2n-2\lambda)+1}}\right\}^{m+1},$$

or, what is the same,
$$|Q(z)| \geqslant \left\{\binom{2n}{2\lambda}^{-1}\binom{n}{\lambda}\binom{2n}{n}\,n!\,2^{-2n-1}\right\}^{m+1}.$$

Put, for the moment,
$$q_\lambda = \binom{2n}{2\lambda}^{-1}\binom{n}{\lambda}\binom{2n}{n} \quad (\lambda = 0, 1, \ldots, n).$$

Then
$$q_\lambda = q_{n-\lambda} \quad\text{and}\quad \frac{q_{\lambda+1}}{q_\lambda} = \frac{2\lambda+1}{2n-2\lambda-1}\begin{cases} <1 & \text{if } \lambda < \dfrac{n-1}{2}, \\ \geqslant 1 & \text{if } \lambda \geqslant \dfrac{n-1}{2}. \end{cases}$$

Therefore
$$q_0 > q_1 > \ldots > q_{[\frac{1}{2}n]} \quad\text{and}\quad q_n > q_{n-1} > \ldots \geqslant q_{[\frac{1}{2}n]},$$

whence
$$\min(q_0, q_1, \ldots, q_n) = q_{[\frac{1}{2}n]} = \binom{2n}{2[\frac{1}{2}n]}^{-1}\binom{n}{[\frac{1}{2}n]}\binom{2n}{n}.$$

Further, since
$$\binom{n}{\lambda} \leqslant \binom{n}{[\frac{1}{2}n]}, \quad \binom{2n}{2\lambda} \leqslant \binom{2n}{n}$$

and
$$\binom{n}{[\frac{1}{2}n]} \geqslant \frac{1}{n+1}\sum_{\lambda=0}^{n}\binom{n}{\lambda} = \frac{2^n}{n+1}$$

for all $\lambda = 0, 1, \ldots, n$, we have

$$q_\lambda \geqslant q_{[\frac{1}{2}n]} \geqslant \binom{n}{[\frac{1}{2}n]} \geqslant \frac{2^n}{n+1}.$$

Hence, when z is on the contour C_λ,

$$|Q(z)| \geqslant \left\{\frac{2^n}{n+1} n!\, 2^{-2n-1}\right\}^{m+1} = (n+1)^{-m-1}\, 2^{-(m+1)(n+1)} (n!)^{m+1}.$$

Therefore, from the integral,

$$|\gamma_\kappa^{(\lambda,h)}| < \frac{1}{2\pi}\frac{2\pi}{2}(n+1)^m \{(n+1)^{-m-1}\, 2^{-(m+1)(n+1)}(n!)^{m+1}\}^{-1} = \tfrac{1}{2}(n+1)^{2m+1}\, 2^{(m+1)(n+1)}(n!)^{-m-1},$$

and so
$$\left|\frac{P}{k!}\gamma_{-k-1}^{(\lambda,h)}\right| = \left|\frac{m!}{k!}N^m (n!)^{m+1}\gamma_{-k-1}^{(\lambda,h)}\right| < \tfrac{1}{2}m!\, N^m (n+1)^{2m+1}\, 2^{(m+1)(n+1)},$$

whence, finally,

$$|A_{hk}(x)| < \tfrac{1}{2}m!\, N^m (n+1)^{2m+1}\, 2^{(m+1)(n+1)} \quad (h, k = 0, 1, \ldots, m).$$

In the notation of majorants, this may also be written as

$$A_{hk}(x) \ll \tfrac{1}{2}m!\, N^m (n+1)^{2m+1}\, 2^{(m+1)(n+1)}(1+x+\ldots+x^n).$$

8. We conclude this chapter by determining an upper bound for $R_h(x)$. In the integral for this function,

$$R_h(x) = \frac{P}{2\pi i}\int_C \frac{z^h x^{z+n}\,dz}{Q(z)},$$

C was assumed to be a circle in the z-plane of centre $z=0$ and arbitrary radius $\rho > n$, described in the positive direction. In order to simplify the final result, we assume, from now on, that $x \neq 1$ and

$$m+1 \geqslant 2|\ln x|,$$

and we fix ρ by
$$\rho = \frac{(m+1)\,n}{|\ln x|},$$

so that
$$\rho \geqslant 2n > n \quad \text{and} \quad \rho - n \geqslant \tfrac{1}{2}\rho.$$

On the contour C,
$$|z^h| \leqslant \rho^m, \quad |x^{z+n}| \leqslant e^{(\rho+n)|\ln x|},$$

since $0 \leqslant h \leqslant m$. Further, on this contour,

$$|Q(z)| = \left|z^{n+1}\left(1+\frac{1}{z}\right)\left(1+\frac{2}{z}\right)\ldots\left(1+\frac{n}{z}\right)\right|^{m+1} \geqslant \rho^{(m+1)(n+1)}\left\{\left(1-\frac{1}{\rho}\right)\left(1-\frac{2}{\rho}\right)\ldots\left(1-\frac{n}{\rho}\right)\right\}^{m+1}.$$

Since
$$\left(1-\frac{1}{\rho}\right)\left(1-\frac{2}{\rho}\right)\ldots\left(1-\frac{n}{\rho}\right) = \left\{\left(1+\frac{1}{\rho-1}\right)\left(1+\frac{2}{\rho-2}\right)\ldots\left(1+\frac{n}{\rho-n}\right)\right\}^{-1}$$

and
$$\left(1+\frac{1}{\rho-1}\right)\left(1+\frac{2}{\rho-2}\right)\ldots\left(1+\frac{n}{\rho-n}\right) \leqslant \exp\left(\sum_{\lambda=1}^{n}\frac{\lambda}{\rho-\lambda}\right) \leqslant \exp\left(\sum_{\lambda=1}^{n}\frac{\lambda}{\rho-n}\right)$$
$$= \exp\left(\frac{n(n+1)}{2(\rho-n)}\right) \leqslant \exp\left(\frac{n(n+1)}{\rho}\right),$$

$Q(z)$ admits on C the lower bound

$$|Q(z)| \geqslant \rho^{(m+1)(n+1)} \exp\left(-\frac{n}{\rho}(m+1)\,(n+1)\right).$$

It is therefore obvious from the integral that

$$|R_h(x)| \leqslant \frac{P}{2\pi} 2\pi\rho\,\rho^m\, e^{(\rho+n)|\ln x|}\left\{\rho^{(m+1)(n+1)}\exp\left(-\frac{n}{\rho}(m+1)\,(n+1)\right)\right\}^{-1}.$$

Here
$$\exp\left(-\frac{n}{\rho}(m+1)(n+1)\right) = e^{-(n+1)|\ln x|},$$
whence
$$|R_h(x)| \leqslant P\rho^{m+1} e^{(\rho+n)|\ln x|} \rho^{-(m+1)(n+1)} e^{(n+1)|\ln x|} = P e^{(2n+1)|\ln x|} e^{\rho|\ln x|} \rho^{-(m+1)n}.$$

On replacing now P and ρ by their expressions in m, n and x, we obtain the following upper bound for $R_h(x)$:
$$|R_h(x)| \leqslant m!\, N^m (n!)^{m+1} e^{(2n+1)|\ln x|} e^{(m+1)n} \left(\frac{(m+1)n}{|\ln x|}\right)^{-(m+1)n},$$
that is,
$$|R_h(x)| \leqslant m!\, N^m (n!)^{m+1} e^{(2n+1)|\ln x|} \left(\frac{e|\ln x|}{(m+1)n}\right)^{(m+1)n}.$$

9. The two inequalities
$$A_{hk}(x) \leqslant \tfrac{1}{2} m!\, N^m (n+1)^{2m+1} 2^{(m+1)(n+1)} (1+x+\ldots+x^n)$$
and
$$|R_h(x)| \leqslant m!\, N^m (n!)^{m+1} e^{(2n+1)|\ln x|} \left(\frac{e|\ln x|}{(m+1)n}\right)^{(m+1)n},$$
proved in the last section, can be put into a more convenient form, if we make use of the elementary inequality*
$$n! \leqslant e n^{\frac{1}{2}} n^n e^{-n}$$
and of the inequality of Rosser†
$$N = [1, 2, \ldots, n] < 2^{\frac{3}{2}n},$$
both of which hold for all positive integers.

The inequality for $|A_{hk}(x)|$ takes then the form
$$A_{hk}(x) \leqslant \tfrac{1}{2} m!\, 2^{\frac{3}{2}mn} (n+1)^{2m+1} 2^{(m+1)(n+1)} (1+x+\ldots+x^n),$$
which may also be written as
$$A_{hk}(x) \leqslant m!\, 2^{m-\frac{3}{2}n} (n+1)^{2m+1} (\surd 32)^{(m+1)n} (1+x+\ldots+x^n).$$

Similarly, the upper bound for $R_h(x)$ becomes
$$|R_h(x)| \leqslant m!\, 2^{\frac{3}{2}mn} (e\surd n)^{m+1} n^{(m+1)n} e^{-(m+1)n} e^{(2n+1)|\ln x|} \left(\frac{e|\ln x|}{(m+1)n}\right)^{(m+1)n},$$
and this may be put in the form
$$|R_h(x)| \leqslant m!\, 2^{-\frac{3}{2}n} (e\surd n)^{m+1} e^{(2n+1)|\ln x|} \left(\frac{8^{\frac{1}{2}}|\ln x|}{m+1}\right)^{(m+1)n}.$$

10. The main results of this chapter may now be formulated as follows.

THEOREM 1. *Let x be a real or complex number different from 0 and 1; let $\ln x$ be the principal value of the logarithm; and let m and n be two positive integers of which the first one satisfies the inequality*
$$m+1 \geqslant 2|\ln x|.$$

* The sequence $a_1, a_2, a_3, \ldots$ defined by
$$a_n = n!\, n^{-(n+\frac{1}{2})} e^n \quad (n = 1, 2, 3, \ldots)$$
is easily seen to be decreasing; therefore $a_n \leqslant a_1 = e$.

† In his paper (1941) Rosser gives the result that $(\ln N)/n$ assumes its maximum at $n = 113$ and that this maximum is less than 1·0389. On the other hand, $\frac{3}{2}\ln 2$ is greater than 1·0397.

Then there exist $(m+1)^2$ *polynomials*

$$A_{hk}(x) \quad (h, k = 0, 1, \ldots, m)$$

in x of degree not greater than n, with the following properties:

(*a*) *The determinant*

$$|A_{hk}(x)|_{h,k=0,1,\ldots,m}$$

does not vanish.

(*b*) *Every polynomial* $A_{hk}(x)$ *has integral coefficients such that*

$$A_{hk}(x) \ll m!\, 2^{m-\frac{3}{2}n}(n+1)^{2m+1}(\sqrt{32})^{(m+1)n}(1+x+\ldots+x^n).$$

(*c*) *The* $m+1$ *functions*

$$R_h(x) = \sum_{k=0}^{m} A_{hk}(x)(\ln x)^k \quad (h = 0, 1, \ldots, m)$$

satisfy the inequalities

$$|R_h(x)| \leqslant m!\, 2^{-\frac{3}{2}n}(\mathrm{e}\sqrt{n})^{m+1}\,\mathrm{e}^{(2n+1)|\ln x|}\left(\frac{8^{\frac{1}{2}}|\ln x|}{m+1}\right)^{(m+1)n}$$

Chapter 2. The logarithms of algebraic numbers

11. The next investigations make use of the following lemma:

Theorem 2. *Let* $f(x) = f_0 + f_1 x + \ldots + f_\phi x^\phi$, *where* $f_\phi > 0$,
be an irreducible polynomial with integral coefficients, and let

$$\xi, \xi_1, \ldots, \xi_{\phi-1}$$

be the roots of the equation $f(x) = 0$. *If*

$$g(x) = g_0 + g_1 x + \ldots + g_\psi x^\psi$$

is a polynomial with integral coefficients for which $g(\xi) \neq 0$, *then*

$$|g(\xi)| \geqslant \{(\phi+1)^\psi\, 3^{\phi\psi}\, \overline{|f(x)|}^{\psi}\, \overline{|g(x)|}^{\phi-1}\}^{-1}.$$

Proof. The hypothesis $g(\xi) \neq 0$ implies that also

$$g(\xi_1) \neq 0, \quad \ldots, \quad g(\xi_{\phi-1}) \neq 0,$$

hence that the product $\gamma = f_\phi^\psi g(\xi)\, g(\xi_1) \ldots g(\xi_{\phi-1})$

does not vanish. This product is symmetrical in $\xi, \xi_1, \ldots, \xi_{\phi-1}$ and of degree ψ in each of these roots. It is therefore a rational integer, whence

$$|\gamma| \geqslant 1.$$

Since, for $l = 1, 2, \ldots, \phi-1$,

$$|g(\xi_l)| \leqslant \overline{|g(x)|}\,(1+|\xi_l|+\ldots+|\xi_l|^\psi) \leqslant \overline{|g(x)|}\,(1+|\xi_l|)^\psi,$$

γ admits the upper bound

$$|\gamma| \leqslant f_\phi^\psi\, |g(\xi)|\, \overline{|g(x)|}^{\phi-1}\left(\prod_{l=1}^{\phi-1}(1+|\xi_l|)\right)^\psi.$$

Now, in the equation for the ξ's,

$$\frac{f_0}{f_\phi} + \frac{f_1}{f_\phi}x + \ldots + x^\phi = 0,$$

the coefficients are in absolute value not greater than

$$\frac{\overline{|f(x)|}}{f_\phi}.$$

Therefore, by a result of Siegel*,

$$(1+|\xi|)\prod_{l=1}^{\phi-1}(1+|\xi_l|)\leqslant(\phi+1)3^{\phi}\frac{\overline{|f(x)|}}{f_\phi},$$

and so even more
$$\prod_{l=1}^{\phi-1}(1+|\xi_l|)\leqslant(\phi+1)\,3^{\phi}\frac{\overline{|f(x)|}}{f_\phi}.$$

Hence
$$1\leqslant|\gamma|\leqslant f_\phi^{\psi}|g(\xi)|\,\overline{|g(x)|}^{\phi-1}\left((\phi+1)\,3^{\phi}\frac{\overline{|f(x)|}}{f_\phi}\right)^{\psi},$$

whence the assertion.

12. Let ξ be a real or complex algebraic number different from 0 and 1, and let

$$f(x)=f_0+f_1x+\ldots+f_\phi x^\phi,\quad \text{where}\quad f_\phi>0,$$

be an irreducible polynomial with integral coefficients of which ξ is a zero. Denote by

$$\eta=\ln\xi$$

the principal value of the logarithm of ξ as defined in § 1. We consider a linear form

$$r=a_0+a_1\eta+\ldots+a_\mu\eta^\mu$$

in the $\mu+1$ powers $1,\eta,\eta^2,\ldots,\eta^\mu$ of η with integral coefficients $a_0,a_1,\ldots,a_\mu$ not all zero. Our aim is to obtain a lower bound for $|r|$ in terms of ξ, the degree μ, and the height

$$a=\max(|a_0|,|a_1|,\ldots,|a_\mu|)$$

of r.

As in the first chapter, let m and n be two positive integers; they will be fixed later, but we assume from now on that
$$m\geqslant\mu,\quad m+1\geqslant2|\eta|.$$

The $m-\mu+1$ linear forms in $1,\eta,\eta^2,\ldots,\eta^m$ derived from r:

$$\begin{aligned}
r&=a_0+a_1\eta+a_2\eta^2+\ldots+a_\mu\eta^\mu,\\
r\eta&=\quad a_0\eta+a_1\eta^2+\ldots+a_{\mu-1}\eta^\mu+a_\mu\eta^{\mu+1},\\
&\vdots\qquad\qquad\ddots\qquad\qquad\qquad\ddots\\
r\eta^{m-\mu}&=\qquad\qquad a_0\eta^{m-\mu}+a_1\eta^{m-\mu+1}+\ldots+a_\mu\eta^m,
\end{aligned}$$

are linearly independent because the matrix Ω of their coefficients contains a non-zero minor of order $m-\mu+1$. For let ν be the largest number for which $a_\nu\neq0$; then the minor of Ω which has a_ν as its upper left-hand corner element is triangular and no elements in its main diagonal vanish.

By the first chapter, the $m+1$ linear forms

$$R_h(\xi)=A_{h0}(\xi)+A_{h1}(\xi)\,\eta+\ldots+A_{hm}(\xi)\,\eta^m\quad(h=0,1,\ldots,m)$$

in $1,\eta,\eta^2,\ldots,\eta^m$ are likewise independent because their determinant is not zero. It is then possible to select μ of these forms, the forms

$$R_{h_1}(\xi),\quad R_{h_2}(\xi),\quad\ldots,\quad R_{h_\mu}(\xi),$$

say, where
$$1\leqslant h_1<h_2<\ldots<h_\mu\leqslant m,$$

such that the $m+1$ linear forms

$$r,\quad r\eta,\quad\ldots,\quad r\eta^{m-\mu},\quad R_{h_1}(\xi),\quad R_{h_2}(\xi),\quad\ldots,\quad R_{h_\mu}(\xi)$$

* Compare the proof of Hilfssatz I in Siegel's paper (1921).

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are also independent. Hence, if $\Delta(x)$ denotes the determinant

$$\Delta(x)=\begin{vmatrix} a_0 & a_1 & a_2 & \dots & \dots & \dots & a_\mu & \dots & \dots & 0 \\ 0 & a_0 & a_1 & \dots & \dots & \dots & a_{\mu-1} & a_\mu & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & & & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_0 & a_1 & \dots & \dots & \dots & a_\mu \\ A_{h_1 0}(x) & A_{h_1 1}(x) & A_{h_1 2}(x) & \dots & \dots & \dots & \dots & \dots & \dots & A_{h_1 m}(x) \\ \vdots & \vdots & \vdots & & & & & & & \vdots \\ A_{h_\mu 0}(x) & A_{h_\mu 1}(x) & A_{h_\mu 2}(x) & \dots & \dots & \dots & \dots & \dots & \dots & A_{h_\mu m}(x) \end{vmatrix},$$

then $\Delta(\xi)$ is the determinant of these linear forms, and therefore

$$\Delta(\xi) \neq 0.$$

13. The first $m-\mu+1$ rows of $\Delta(x)$ consist of integers, while the last μ rows are formed by polynomials in x at most of degree n and with integral coefficients. Hence $\Delta(x)$ is itself a polynomial in x of degree not greater than μn with integral coefficients. An upper bound for these coefficients is obtained by the following estimation.

By theorem 1,

$$A_{hk}(x) \ll A(1+x+\dots+x^n), \quad \text{where} \quad A = m!\,2^{m-\frac{1}{2}n}(n+1)^{2m+1}(\surd 32)^{(m+1)n}.$$

Therefore, the product of any μ of the polynomials $A_{hk}(x)$ is majorized by

$$A^\mu(1+x+\dots+x^n)^\mu.$$

Here $(1+x+\dots+x^n)^\mu$ can be written as

$$(1+x+\dots+x^n)^\mu = j_0+j_1x+\dots+j_{\mu n}x^{\mu n},$$

where the j's are integers and positive. On putting $x=1$, we see that

$$j_0+j_1+\dots+j_{\mu n} = (n+1)^\mu,$$

and so $$(1+x+\dots+x^n)^\mu \ll (n+1)^\mu(1+x+\dots+x^{\mu n}).$$

Now $\Delta(x)$ consists of $(m+1)!$ terms, each of which is clearly majorized by the expression

$$a^{m-\mu+1}A^\mu(1+x+\dots+x^n)^\mu.$$

Hence $$\Delta(x) \ll (m+1)!\,a^{m-\mu+1}A^\mu(n+1)^\mu(1+x+\dots+x^{\mu n}),$$

or, say, $$\Delta(x) \ll A_1 a^{m-\mu+1}(1+x+\dots+x^{\mu n}),$$

where $A_1 = (m+1)!\,A^\mu(n+1)^\mu = (m+1)\,(m!)^{\mu+1}\,2^{(m-\frac{1}{2}n)\mu}(n+1)^{2(m+1)\mu}(\surd 32)^{(m+1)\mu n}$.

In just the same way, we can majorize the cofactors of the elements of $\Delta(x)$. In particular, denote by $$\Phi_0(x),\quad \Phi_1(x),\quad \dots,\quad \Phi_{m-\mu}(x),\quad \Psi_1(x),\quad \dots,\quad \Psi_\mu(x)$$

the cofactors of the $m+1$ successive elements of the first column of $\Delta(x)$. A similar calculation to the last leads to

$$\Phi_i(x) \ll A_2 a^{m-\mu}(1+x+\dots+x^{\mu n}) \quad (i=0,1,\dots,m-\mu),$$

where, for shortness,

$$A_2 = (m!)^{\mu+1}\,2^{(m-\frac{1}{2}n)\mu}(n+1)^{2(m+1)\mu}(\surd 32)^{(m+1)\mu n},$$

and also to $$\Psi_j(x) \ll A_3 a^{m-\mu+1}(1+x+\dots+x^{(\mu-1)n}) \quad (j=1,2,\dots,\mu),$$

where $$A_3 = (m!)^\mu\,2^{(m-\frac{1}{2}n)(\mu-1)}(n+1)^{2(m+1)(\mu-1)}(\surd 32)^{(m+1)(\mu-1)n}.$$

The determinant $\Delta(x)$ can be expressed in terms of these cofactors. Multiply the second, third, ..., $(m+1)$st column of $\Delta(x)$ by $\eta, \eta^2, \ldots, \eta^m$, respectively, and add to the first column. The new first column is then

$$r, \quad r\eta, \quad \ldots, \quad r\eta^{m-\mu}, \quad R_{h_1}(x), \quad R_{h_2}(x), \quad \ldots, \quad R_{h_\mu}(x),$$

and therefore, identically in x,

$$\Delta(x) = r\sum_{i=0}^{m-\mu} \eta^i \Phi_i(x) + \sum_{j=1}^{\mu} R_{h_j}(x)\, \Psi_j(x).$$

14. Put now $x = \xi$ in the last identity and write

$$f = \overline{|f(x)|} = \max(|f_0|, |f_1|, \ldots, |f_\phi|).$$

Then, first, from theorem 2,

$$|\Delta(\xi)| \geqslant \{(\phi+1)^{\mu n} 3^{\mu\phi n} f^{\mu n} (A_1 a^{m-\mu+1})^{\phi-1}\}^{-1},$$

or

$$|\Delta(\xi)| \geqslant A_4^{-1} a^{-(m-\mu+1)(\phi-1)},$$

where

$$A_4 = (\phi+1)^{\mu n} 3^{\mu\phi n} f^{\mu n} (m+1)^{\phi-1} (m!)^{(\mu+1)(\phi-1)} 2^{(m-\frac{3}{2}n)\mu(\phi-1)} (n+1)^{2(m+1)\mu(\phi-1)} (\sqrt{32})^{(m+1)\mu(\phi-1)n}.$$

Secondly,

$$|\Phi_i(\xi)| \leqslant A_2 a^{m-\mu}(1+|\xi|+\ldots+|\xi|^{\mu n}),$$

$$|\Psi_j(\xi)| \leqslant A_3 a^{m-\mu+1}(1+|\xi|+\ldots+|\xi|^{(\mu-1)n}).$$

Here

$$\frac{1}{f+1} \leqslant |\xi| \leqslant f+1,$$

because the equation

$$f_0 + f_1\xi + \ldots + f_\phi \xi^\phi = 0$$

for ξ may be written as

$$f_\phi \xi = -\left(\frac{f_{\phi-1}}{\xi} + \frac{f_{\phi-2}}{\xi^2} + \ldots + \frac{f_0}{\xi^{\phi-1}}\right),$$

and so either $|\xi| \leqslant 1$ or

$$|\xi| \leqslant |f_\phi \xi| \leqslant f(|\xi|^{-1} + |\xi|^{-2} + \ldots) = f\frac{|\xi|}{|\xi|-1}, \quad \text{whence} \quad |\xi| \leqslant f+1.$$

A similar proof holds for the lower bound. Therefore, since $f+1 \leqslant 2f$,

$$1+|\xi|+\ldots+|\xi|^{\mu n} \leqslant 1+(f+1)+\ldots+(f+1)^{\mu n} = \frac{(f+1)^{\mu n+1}-1}{f} < 2^{\mu n+1} f^{\mu n},$$

and, in the same way, $1+|\xi|+\ldots+|\xi|^{(\mu-1)n} \leqslant 2^{(\mu-1)n+1} f^{(\mu-1)n}$.

Hence the inequalities for $\Phi_i(\xi)$ and $\Psi_j(\xi)$ take the simpler form,

$$|\Phi_i(\xi)| \leqslant 2^{\mu n+1} f^{\mu n} A_2 a^{m-\mu},$$

$$|\Psi_j(\xi)| \leqslant 2^{(\mu-1)n+1} f^{(\mu-1)n} A_3 a^{m-\mu+1}.$$

Thirdly, the logarithm $\eta = \ln\xi$ was defined as the principal value, so that by the bounds for ξ,

$$|\eta| \leqslant \{(\ln|\xi|)^2+\pi^2\}^{\frac{1}{2}} \leqslant \{(\ln(f+1))^2+\pi^2\}^{\frac{1}{2}}.$$

Since $\ln(f+1) \leqslant f$, this means that

$$|\eta| \leqslant \{f^2+\pi^2\}^{\frac{1}{2}} \leqslant f+3 \leqslant 4f.$$

Since
$$\left\{1+\left(\frac{\pi}{\ln 2}\right)^2\right\}^{\frac{1}{2}} < 10^{\frac{2}{3}},$$
η admits the further bound given by
$$|\eta| \leqslant \left(\ln(f+1)\right)\left\{1+\left(\frac{\pi}{\ln(f+1)}\right)^2\right\}^{\frac{1}{2}} \leqslant \left(\ln(f+1)\right)\left\{1+\left(\frac{\pi}{\ln 2}\right)^2\right\}^{\frac{1}{2}} \leqslant 10^{\frac{2}{3}}\ln(f+1).$$
The first of these two estimations implies, in particular, that
$$\sum_{i=0}^{m-\mu} |\eta|^i \leqslant \sum_{i=0}^{m-\mu} (f+3)^i = \frac{(f+3)^{m-\mu+1}-1}{f+2} \leqslant 2(f+3)^{m-\mu} \leqslant 2^{2(m-\mu)+1} f^{m-\mu},$$
because $f+3<2(f+2)$. Therefore, by the bound for $\Phi_i(\xi)$,
$$\left|\sum_{i=0}^{m-\mu} \eta^i \Phi_i(\xi)\right| \leqslant A_5\, a^{m-\mu},$$
where, for shortness, we have put
$$A_5 = 2^{2m+\mu(n-2)+2} f^{m+\mu(n-1)} A_2,$$
that is,
$$A_5 = (m!)^{\mu+1}\, 2^{m(\mu+2)-\mu(\frac{1}{2}n+2)+2} f^{m+\mu(n-1)} (n+1)^{2(m+1)\mu} (\sqrt{32})^{(m+1)\mu n}.$$

Next, by theorem 1,
$$|R_h(\xi)| \leqslant m!\, 2^{-\frac{3}{2}n} (\mathrm{e}\sqrt{n})^{m+1}\, \mathrm{e}^{(2n+1)|\eta|} \left(\frac{\sqrt{(8)}\,|\eta|}{m+1}\right)^{(m+1)n}.$$
The two estimates for η imply then that
$$|R_h(\xi)| \leqslant m!\, 2^{-\frac{3}{2}n} (\mathrm{e}\sqrt{n})^{m+1}\, \mathrm{e}^{4f(2n+1)} \left(\frac{8^{\frac{1}{2}}\, 10^{\frac{2}{3}} \ln(f+1)}{m+1}\right)^{(m+1)n},$$
hence that
$$\left|\sum_{j=1}^{\mu} R_{h_j}(\xi)\, \Psi_j(\xi)\right| \leqslant A_6\, a^{m-\mu+1},$$
with the abbreviation
$$A_6 = \mu\,.\,m!\, 2^{-\frac{3}{2}n} (\mathrm{e}\sqrt{n})^{m+1}\, \mathrm{e}^{4f(2n+1)} \left(\frac{8^{\frac{1}{2}}\, 10^{\frac{2}{3}} \ln(f+1)}{m+1}\right)^{(m+1)n} 2^{(\mu-1)n+1} f^{(\mu-1)n} A_3.$$
In explicit form,
$$A_6 = \mu\,.\,m!\, 2^{-\frac{3}{2}n} (\mathrm{e}\sqrt{n})^{m+1}\, \mathrm{e}^{4f(2n+1)} \left(\frac{8^{\frac{1}{2}}\, 10^{\frac{2}{3}} \ln(f+1)}{m+1}\right)^{(m+1)n}$$
$$\times 2^{(\mu-1)n+1} f^{(\mu-1)n} (m!)^{\mu}\, 2^{(m-\frac{3}{2}n)(\mu-1)} (n+1)^{2(m+1)(\mu-1)} (\sqrt{32})^{(m+1)(\mu-1)n},$$
or, after some simplification,
$$A_6 = \mu(m!)^{\mu+1}\, 2^{m(\mu-1)-\frac{1}{2}n(\mu+2)+1} (\mathrm{e}\sqrt{n})^{m+1}\, \mathrm{e}^{4f(2n+1)} f^{(\mu-1)n}$$
$$\times (n+1)^{2(m+1)(\mu-1)} \left(\frac{2^{\frac{13}{6}+\frac{5}{2}(\mu-1)}\, 5^{\frac{2}{3}} \ln(f+1)}{m+1}\right)^{(m+1)n}.$$

The equation
$$\Delta(\xi) = r\sum_{i=0}^{m-\mu} \eta^i \Phi_i(\xi) + \sum_{j=1}^{\mu} R_{h_j}(\xi)\, \Psi_j(\xi)$$
leads therefore finally to the inequality
$$A_4^{-1}\, a^{-(m-\mu+1)(\phi-1)} \leqslant A_5\, a^{m-\mu}\, |r| + A_6\, a^{m-\mu+1},$$
that is,
$$1 \leqslant A_4 A_5\, a^{(m-\mu+1)\phi-1}\, |r| + A_4 A_6\, a^{(m-\mu+1)\phi}.$$

Here, after some simplification,

$$A_4A_5 = (\phi+1)^{\mu n}\, 3^{\mu\phi n} f^{m+\mu(2n-1)}(m+1)^{\phi-1}\,(m!)^{(\mu+1)\phi}\,(n+1)^{2(m+1)\mu\phi}$$
$$\times\, 2^{(m-\frac{3}{2}n)\mu\phi+\mu(n-2)+2(m+1)}\,(\sqrt{32})^{(m+1)\mu\phi n}$$

and

$$A_4A_6 = \mu(\phi+1)^{\mu n}\, 3^{\mu\phi n} f^{(2\mu-1)n}(m+1)^{\phi-1}\,(m!)^{(\mu+1)\phi}\,(n+1)^{2(m+1)(\mu\phi-1)}$$
$$\times\,(\mathrm{e}\sqrt{n})^{m+1}\,\mathrm{e}^{4f(2n+1)}\,2^{(m-\frac{3}{2}n)\mu\phi+\mu n-m-n+1}\left(\frac{2^{-\frac{1}{3}+\frac{5}{2}\mu\phi}\,5^{\frac{2}{3}}\ln(f+1)}{m+1}\right)^{(m+1)n}.$$

15. Assume from now on that

$$m+1 \geqslant 50f \geqslant 50, \quad n \geqslant 30\ln(m+1).$$

Then, first, by $\ln(f+1) \leqslant f$,

$$|\eta| \leqslant 10^{\frac{2}{3}}\ln(f+1) < 10f < \tfrac{1}{2}(m+1)$$

and so the condition

$$m+1 \geqslant 2\,|\ln\xi|$$

from theorem 1 is automatically satisfied. Secondly,

$$n \geqslant 30\ln(m+1) \geqslant 30\ln 50 > 100$$

and also

$$n > 30\ln f.$$

Thirdly, if t assumes all positive values, then $(\ln t)/t$ assumes its maximum at $t=\mathrm{e}$, and therefore

$$\frac{\ln t}{t} \leqslant \frac{1}{\mathrm{e}}.$$

Since $\mu \geqslant 1$, $\phi \geqslant 1$, we obtain then the following estimates:

$$\{\mu\}^{\frac{1}{\mu\phi(m+1)n}} \leqslant \mathrm{e}^{\frac{\ln\mu}{\mu}\frac{1}{(m+1)n}} \leqslant \mathrm{e}^{\frac{1}{\mathrm{e}}\frac{1}{5000}},$$

$$\{(\phi+1)^{\mu n}\,3^{\mu\phi n}\}^{\frac{1}{\mu\phi(m+1)n}} \leqslant \{6^{\mu\phi n}\}^{\frac{1}{\mu\phi(m+1)n}} = 6^{\frac{1}{m+1}} \leqslant 6^{\frac{1}{50}},$$

$$\{f^{m+\mu(2n-1)}\}^{\frac{1}{\mu\phi(m+1)n}} \leqslant f^{\frac{1}{n}+\frac{2}{m+1}} = \mathrm{e}^{\frac{\ln f}{n}+\frac{2\ln f}{f}\frac{f}{m+1}} \leqslant \mathrm{e}^{\frac{1}{30}+\frac{1}{\mathrm{e}}\frac{1}{25}},$$

$$\{f^{(2\mu-1)n}\}^{\frac{1}{\mu\phi(m+1)n}} \leqslant f^{\frac{2}{m+1}} \leqslant \mathrm{e}^{\frac{1}{\mathrm{e}}\frac{1}{25}},$$

$$\{(m+1)^{\phi-1}\}^{\frac{1}{\mu\phi(m+1)n}} \leqslant \mathrm{e}^{\frac{\ln(m+1)}{(m+1)n}} \leqslant \mathrm{e}^{\frac{1}{50\times 30}} = \mathrm{e}^{\frac{1}{1500}},$$

$$\{(m!)^{(\mu+1)\phi}\}^{\frac{1}{\mu\phi(m+1)n}} \leqslant \{(m^m)^{(\mu+1)\phi}\}^{\frac{1}{\mu\phi(m+1)n}} < \mathrm{e}^{\frac{2\ln(m+1)}{n}} \leqslant \mathrm{e}^{\frac{1}{15}},$$

$$\{(n+1)^{2(m+1)\mu\phi}\}^{\frac{1}{\mu\phi(m+1)n}} = \mathrm{e}^{\frac{2\ln(n+1)}{n}} \leqslant \mathrm{e}^{\frac{2\ln 101}{100}},$$

$$\{(n+1)^{2(m+1)(\mu\phi-1)}\}^{\frac{1}{\mu\phi(m+1)n}} \leqslant \mathrm{e}^{\frac{2\ln 101}{100}},$$

$$\{(\mathrm{e}\sqrt{n})^{m+1}\}^{\frac{1}{\mu\phi(m+1)n}} \leqslant \mathrm{e}^{\frac{1}{n}+\frac{1}{2}\frac{\ln n}{n}} \leqslant \mathrm{e}^{\frac{1}{100}+\frac{\ln 100}{100}},$$

$$\{\mathrm{e}^{4f(2n+1)}\}^{\frac{1}{\mu\phi(m+1)n}} \leqslant \mathrm{e}^{\frac{12fn}{(m+1)n}} \leqslant \mathrm{e}^{\frac{6}{25}},$$

$$\{2^{(m-\frac{3}{2}n)\mu\phi+\mu(n-2)+2(m+1)}\}^{\frac{1}{\mu\phi(m+1)n}} \leqslant \{2^{m\mu\phi+2(m+1)}\}^{\frac{1}{\mu\phi(m+1)n}} \leqslant 2^{\frac{3}{n}} \leqslant 2^{\frac{3}{100}},$$

$$\{2^{(m-\frac{3}{2}n)\mu\phi+\mu n-m-n+1}\}^{\frac{1}{\mu\phi(m+1)n}} \leqslant \{2^{m\mu\phi}\}^{\frac{1}{\mu\phi(m+1)n}} \leqslant 2^{\frac{1}{n}} \leqslant 2^{\frac{1}{100}}.$$

These estimates imply that

$$A_4 A_5 \leqslant \left(6^{\frac{1}{50}} \mathrm{e}^{\frac{1}{30}+\frac{1}{25\mathrm{e}}} \mathrm{e}^{\frac{1}{1500}} \mathrm{e}^{\frac{1}{15}} \mathrm{e}^{\frac{2\ln 101}{100}} 2^{\frac{3}{100}} \sqrt{32}\right)^{\mu\phi(m+1)n}$$

and

$$A_4 A_6 \leqslant \left(\mathrm{e}^{\frac{1}{5000\mathrm{e}}} 6^{\frac{1}{50}} \mathrm{e}^{\frac{1}{25\mathrm{e}}} \mathrm{e}^{\frac{1}{1500}} \mathrm{e}^{\frac{1}{15}} \mathrm{e}^{\frac{2\ln 101}{100}} \mathrm{e}^{\frac{1}{100}+\frac{\ln 100}{100}} \mathrm{e}^{\frac{6}{25}} 2^{\frac{1}{100}} \sqrt{32}\right)^{\mu\phi(m+1)n} \left(\frac{2^{-\frac{1}{3}} 5^{\frac{2}{3}} \ln (f+1)}{m+1}\right)^{(m+1)n}.$$

A simple numerical calculation gives then

$$A_4 A_5 < \mathrm{e}^{1\cdot 998\mu\phi(m+1)n}, \quad A_4 A_6 < \mathrm{e}^{2\cdot 247\mu\phi(m+1)n} \left(\frac{\sqrt[3]{(\frac{25}{2})} \ln (f+1)}{m+1}\right)^{(m+1)n},$$

and, *a fortiori*, $\quad A_4 A_5 < \mathrm{e}^{2\mu\phi(m+1)n}, \quad A_4 A_6 < \left(\frac{\sqrt[3]{(\frac{25}{2})}\, \mathrm{e}^{\frac{9}{4}\mu\phi} \ln (f+1)}{m+1}\right)^{(m+1)n}.$

Here, in the second formula, $\quad \sqrt[3]{\frac{25}{2}} < \mathrm{e}, \quad 2 < \mathrm{e}^{\frac{3}{4}}.$

This formula implies therefore that

$$A_4 A_6 < \frac{1}{2} \left(\frac{\mathrm{e}^{3\mu\phi+1} \ln (f+1)}{m+1}\right)^{(m+1)n}.$$

Since both a and μ are positive integers, we find then that

$$A_4 A_5\, a^{(m-\mu+1)\phi-1} < \mathrm{e}^{2\mu\phi(m+1)n}\, a^{(m+1)\phi}$$

and

$$A_4 A_6\, a^{(m-\mu+1)\phi} < \frac{1}{2} \left(\frac{\mathrm{e}^{3\mu\phi+1} \ln (f+1)}{m+1}\right)^{(m+1)n} a^{(m+1)\phi}.$$

Now it was proved in the last section that

$$1 \leqslant A_4 A_5\, a^{(m-\mu+1)\phi-1} \,|\, r \,|\, + A_4 A_6\, a^{(m-\mu+1)\phi},$$

so that, by the formulae just proved, we have

$$1 < \mathrm{e}^{2\mu\phi(m+1)n}\, a^{(m+1)\phi} \,|\, r \,|\, + \frac{1}{2} \left(\frac{\mathrm{e}^{3\mu\phi+1} \ln (f+1)}{m+1}\right)^{(m+1)n} a^{(m+1)\phi}.$$

If here the second term on the right-hand side does not exceed $\frac{1}{2}$, the first term necessarily does so, and a lower bound for r follows at once. We therefore finally choose m by

$$m+1 = \max\left([\mathrm{e}^{4\mu\phi+1} \ln (f+1)]+1,\ 50f\right)$$

and afterwards n by $\quad n = \max\left(30 \ln (m+1), \left[\frac{\ln a}{\mu}\right]+1\right).$

This choice is permitted because the former restrictions on m and n are evidently satisfied. From it,

$$m+1 > \mathrm{e}^{4\mu\phi+1} \ln (f+1), \quad \frac{\mathrm{e}^{3\mu\phi+1} \ln (f+1)}{m+1} < \mathrm{e}^{-\mu\phi} < 1$$

and

$$n > \frac{\ln a}{\mu}, \quad \mathrm{e}^{\mu n} > a.$$

Hence

$$\frac{1}{2} \left(\frac{\mathrm{e}^{3\mu\phi+1} \ln (f+1)}{m+1}\right)^{(m+1)n} a^{(m+1)\phi} < \frac{1}{2} \left(\frac{\mathrm{e}^{3\mu\phi+1} \ln (f+1)}{\mathrm{e}^{4\mu\phi+1} \ln (f+1)}\right)^{(m+1)n} a^{(m+1)\phi} = \frac{1}{2} \left(\frac{a}{\mathrm{e}^{\mu n}}\right)^{(m+1)\phi} < \frac{1}{2},$$

and the second term is in fact less than $\frac{1}{2}$. Therefore, as already said, we find the following lower bound for r:

$$|\, r \,| > \tfrac{1}{2} \left(\mathrm{e}^{2\mu\phi(m+1)n}\, a^{(m+1)\phi}\right)^{-1}.$$

Our discussion has thus led to this general result:

THEOREM 3. *Let* ξ *be a real or complex algebraic number different from* 0 *and* 1, *and let*

$$f_0+f_1x+\ldots+f_\phi x^\phi = 0 \quad (f_\phi>0)$$

be an irreducible equation with integral coefficients for ξ; *write*

$$f = \max(|f_0|, |f_1|, \ldots, |f_\phi|).$$

Denote by
$$\eta = \ln\xi$$
the principal value of the logarithm of ξ, *and by*

$$r = a_0+a_1\eta+\ldots+a_\mu\eta^\mu,$$

where
$$a = \max(|a_0|, |a_1|, \ldots, |a_\mu|) \geqslant 1,$$
a polynomial in η *with integral coefficients not all zero. Put*

$$m = \max([e^{4\mu\phi+1}\ln(f+1)], 50f-1),$$

and
$$n = \max\left(30\ln(m+1), \left[\frac{\ln a}{\mu}\right]+1\right).$$

Then
$$|r| > \tfrac{1}{2}(e^{2\mu n}a)^{-(m+1)\phi},$$
and therefore η *is transcendental.*

Remark. The hypothesis that $\eta = \log\xi$ is the *principal value* of the logarithm is not essential in this theorem, and a similar result can be proved for each other value.

16. Theorem 3 establishes a lower bound for r uniformly in the four parameters f, ϕ, a and μ. On specializing these, we obtain results that are of interest in themselves.

Assume, first, that ξ and η, hence also f and ϕ, are fixed, but that μ is so large that

$$e^{4\mu\phi+1} \geqslant \frac{50f}{\ln(f+1)},$$

and that, with this choice of μ, a satisfies the inequality

$$a \geqslant (m+1)^{30\mu}.$$

Then
$$m+1 = [e^{4\mu\phi+1}\ln(f+1)]+1 \leqslant e^{4\mu\phi+1}\ln(f+1)+1$$

and
$$n = \left[\frac{\ln a}{\mu}\right]+1 \leqslant \frac{\ln a}{\mu}+1,$$

and the bound for r implies that

$$|r| > \tfrac{1}{2}(e^{2\mu}a^3)^{-\{e^{4\mu\phi+1}\ln(f+1)+1\}\phi}.$$

In terms of my old classification of transcendental numbers*, $\eta = \ln\xi$ cannot then be a U-number, but is either an S-number or a T-number, and furthermore

$$\omega_\mu(\eta) \leqslant 3(e^{4\mu\phi+1}\ln(f+1)+1)\phi.$$

There is no difficulty in improving this inequality slightly to

$$\omega_\mu(\eta) = O(e^{\theta\mu\phi}) \quad \text{as} \quad \mu\to\infty;$$

here θ may be any constant greater than $\frac{1}{2}(\ln 32)$.

* See my paper (1932, §1).

The inequality thus proved for $\omega_\mu(\eta)$ generalizes an old result of mine (1932, Satz 5) which had, until now, only been proved for the logarithms of rational numbers.

17. The estimate for r given in theorem 3 is reasonably sharp when a tends to infinity while ϕ, f and μ are fixed; it is very much less good when these last three parameters are also allowed to increase indefinitely.

For assume, as a second application, that both μ and ϕ are unbounded, but that f and a remain fixed. Then, finally,

$$m+1 = [e^{4\mu\phi+1}\ln(f+1)+1], \quad n = 30\ln(m+1)$$

and therefore $$|r| > \tfrac{1}{2}(e^{60\mu\ln[e^{4\mu\phi+1}\ln(f+1)+1]}a)^{-[e^{4\mu\phi+1}\ln(f+1)+1]\phi},$$

whence $$|r| > e^{-O(\mu^2\phi^2 e^{4\mu\phi})}.$$

For constant ϕ, this inequality is contained in one by Fel'dman (1951) that is very much sharper.

As a third application, assume that only f increases while the other three parameters are fixed. Then the theorem leads to $$|r| > e^{-O(f\ln f)}.$$

18. As a final application, let ξ and ζ be two real algebraic numbers satisfying the inequalities $$\xi > 1, \quad \zeta > 0.$$

By Lindemann's theorem, $$\xi^u - e^{v\zeta} \neq 0$$

for any two positive integers u and v. We shall improve this inequality by replacing it by a lower bound for $$|\xi^u - e^{v\zeta}|$$
in terms of u and v.

If, first, $$\xi^u e^{-v\zeta} \leqslant \tfrac{1}{2} \quad \text{or} \quad \xi^u e^{-v\zeta} \geqslant 2,$$

then $$|\xi^u - e^{v\zeta}| \geqslant \tfrac{1}{2}e^{v\zeta} \geqslant \xi^u \quad \text{or} \quad |\xi^u - e^{v\zeta}| \geqslant \tfrac{1}{2}\xi^u \geqslant e^{v\zeta},$$

respectively, hence in either case,

$$|\xi^u - e^{v\zeta}| \geqslant \tfrac{1}{2}\max(\xi^u, e^{v\zeta}).$$

Assume therefore, secondly, that $$\tfrac{1}{2} \leqslant \xi^u e^{-v\zeta} \leqslant 2.$$

Then $$\min(\xi^u, e^{v\zeta}) \geqslant \tfrac{1}{2}\max(\xi^u, e^{v\zeta}),$$

and we deduce from the mean value theorem of the differential calculus that

$$\frac{\xi^u - e^{v\zeta}}{u\ln\xi - v\zeta} \geqslant \min(\xi^u, e^{v\zeta}) \geqslant \tfrac{1}{2}\max(\xi^u, e^{v\zeta}).$$

Let, as before, ξ be a root of the irreducible equation

$$f_0 + f_1 x + \dots + f_\phi x^\phi = 0 \quad (f_\phi > 0)$$

and put $$\eta = \ln\xi, \quad f = \max(|f_0|, |f_1|, \dots, |f_\phi|).$$

Denote further by $$g_0 + g_1 x + \dots + g_\psi x^\psi = 0 \quad (g_\psi > 0)$$

an irreducible equation for ζ with integral coefficients, and write

$$g = \max(|g_0|, |g_1|, \dots, |g_\psi|).$$

The number $\frac{v}{u}\zeta$ then satisfies the equation

$$g_0 v^\psi + g_1 u v^{\psi-1} x + \ldots + g_\psi u^\psi x^\psi = 0.$$

By the result proved in § 16, there exist two positive numbers c_1 and c_2, independent of u and v, such that

$$|g_0 v^\psi + g_1 u v^{\psi-1} \eta + \ldots + g_\psi u^\psi \eta^\psi| \geq c_1 \{\max(u, v)\}^{-c_2};$$

this follows on identifying $\mu, \quad a_0, \quad a_1, \quad \ldots, \quad a_\mu$

with $\psi, \quad g_0 v^\psi, \quad g_1 u v^{\psi-1}, \quad \ldots, \quad g_\psi u^\psi,$

respectively, because then

$$a = \max(|a_0|, |a_1|, \ldots, |a_\mu|) \leq g\{\max(u, v)\}^\psi.$$

Denote by $\zeta, \zeta_1, \ldots, \zeta_{\psi-1}$ the conjugates of ζ, hence by $\frac{v}{u}\zeta, \frac{v}{u}\zeta_1, \ldots, \frac{v}{u}\zeta_{\psi-1}$ the conjugates of $\frac{v}{u}\xi$. Then, identically in x,

$$g_0 v^\psi + g_1 u v^{\psi-1} x + \ldots + g_\psi u^\psi x^\psi = g_\psi (ux - v\zeta)(ux - v\zeta_1) \ldots (ux - v\zeta_{\psi-1}),$$

and the inequality for η may be written as

$$g_\psi |(u\eta - v\zeta)(u\eta - v\zeta_1) \ldots (u\eta - v\zeta_{\psi-1})| \geq c_1 \{\max(u, v)\}^{-c_2}.$$

Here every linear factor $u\eta - v\zeta_1, \quad \ldots, \quad u\eta - v\zeta_{\psi-1}$

is of absolute value not larger than $c_3 \max(u, v)$, where $c_3 > 0$ does not depend on u and v. The last inequality implies therefore that

$$|u\eta - v\zeta| \geq 2c_4 \{\max(u, v)\}^{-c_5},$$

where $c_4 > 0$ and $c_5 > 0$ likewise are independent of u and v, whence

$$|\xi^u - e^{v\zeta}| \geq c_4 \max(\xi^u, e^{v\zeta}) \{\max(u, v)\}^{-c_5}.$$

It follows that there exists to every positive number ϵ a positive number $\gamma = \gamma(\epsilon)$ independent of u and v such that for all positive integral values of these variables,

$$|\xi^u - e^{v\zeta}| \geq \gamma \{\max(\xi^u, e^{v\zeta})\}^{1-\epsilon}.$$

From this inequality we deduce at once that the lower bound

$$M(\xi, e^\zeta) = \inf_{u,v=1,2,3,\ldots} |\xi^u - e^{v\zeta}|$$

is attained and is positive. It is further clear that for any t there are at most a finite number of pairs of positive integers u, v such that

$$|\xi^u - e^{v\zeta}| \leq t.$$

Finally, when t tends to infinity, the number $N(t)$ of integral pairs u, v satisfying this inequality has the property that $N(t) = O((\ln t)^2)$.

CHAPTER 3. THE LOGARITHMS OF RATIONAL NUMBERS

19. In this chapter the estimations that led to theorem 3 are repeated, this time, however, under the restrictive assumption that ξ is a rational number and that only rational approximations of $\ln\xi$ are considered.

Let ξ be the rational number
$$\xi = \frac{f}{f_1},$$
where f and f_1 are two positive integers such that
$$f > f_1 \geqslant 1,$$
so that
$$f \geqslant 2, \quad 1 < \xi \leqslant f, \quad 0 < \ln\xi \leqslant \ln f.$$
Denote by $a = a_0 > 0$, $a_1 \geqslant 0$, ..., $a_m \geqslant 0$ a system of $m+1$ integers and put
$$\lambda_0 = 0, \quad \lambda_k = (\ln\xi)^k - \frac{a_k}{a} \quad (k = 1, 2, \ldots, m).$$
Further, write
$$\lambda = \max(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_m|);$$
our next aim is to determine a lower bound for λ.

20. The identities
$$R_h(\xi) = \sum_{k=0}^{m} A_{hk}(\xi)\,(\ln\xi)^k \quad (h = 0, 1, \ldots, m)$$
lead immediately to the relations
$$\frac{1}{a}\sum_{k=0}^{m} A_{hk}(\xi)\,a_k = R_h(\xi) - \sum_{k=0}^{m} A_{hk}(\xi)\,\lambda_k \quad (h = 0, 1, \ldots, m).$$
Here the sums
$$b_h = f_1^n \sum_{k=0}^{m} A_{hk}(\xi)\,a_k \quad (h = 0, 1, \ldots, m)$$
assume integral values, and since
$$D(\xi) \neq 0,$$
at least one of them is different from zero. There is then an index h such that
$$b_h \neq 0 \quad \text{and therefore} \quad |b_h| \geqslant 1,$$
and so, with this choice of h,
$$(af_1^n)^{-1} \leqslant |R_h(\xi)| + m\lambda \max_{k=0,1,\ldots,m}(|A_{hk}(\xi)|). \tag{1}$$

Denote now by α, β, γ three positive constants to be selected later; in particular, let
$$\alpha \geqslant 2.$$
From now on, we assume that
$$m = [\alpha \ln f] \tag{2}$$
and
$$n \geqslant \max[\beta \ln(m+1), \gamma \ln(n+1), 2]. \tag{3}$$

The condition for m implies that
$$m+1 \geqslant 2\ln f \geqslant 2\ln\xi,$$
so that theorem 1 may be applied; hence
$$|A_{hk}(\xi)| \leqslant m!\,2^{m-\frac{3}{2}n}(n+1)^{2m+1}(\sqrt{32})^{(m+1)n}(1+\xi+\ldots+\xi^n)$$
and
$$|R_h(\xi)| \leqslant m!\,2^{-\frac{3}{2}n}(\mathrm{e}\sqrt{n})^{m+1}\,\mathrm{e}^{(2n+1)\ln\xi}\left(\frac{\sqrt{(8)}\ln\xi}{m+1}\right)^{(m+1)n}.$$

Here, from the inequality in § 9,

$$m! = \frac{(m+1)!}{m+1} \leqslant \frac{(m+1)^{m+1}\,\mathrm{e}^{-m}}{\sqrt{(m+1)}}.$$

Further $$1+\xi+\ldots+\xi^n < (n+1)\,\xi^n$$

and $$f_1^n \xi^n = f^n, \quad f_1^n \xi^{2n+1} \leqslant f^{2n+1}.$$

We find therefore that

$$|A_{hk}(\xi)| < \frac{(m+1)^{m+1}\,\mathrm{e}^{-m}}{\sqrt{(m+1)}}\, 2^{m-\frac{3}{2}n}(n+1)^{2m+2}\,(\sqrt{32})^{(m+1)n}\,\xi^n,$$

whence

$$f_1^n m \max_{k=0,1,\ldots,m} (|A_{hk}(\xi)|) \leqslant \frac{2^{-\frac{3}{2}n} m}{\sqrt{(m+1)}} \left(\frac{2}{\mathrm{e}}\right)^m \{(m+1)\,(n+1)^2\}^{m+1} \sqrt{(32)^{(m+1)n}}\, f^n;$$

and that $$|R_h(\xi)| \leqslant \frac{(m+1)^{m+1}\,\mathrm{e}^{-m}}{\sqrt{(m+1)}}\, 2^{-\frac{3}{2}n} (\mathrm{e}\sqrt{n})^{m+1}\, \xi^{2n+1} \left(\frac{\sqrt{(8)}\ln \xi}{m+1}\right)^{(m+1)n},$$

whence $$f_1^n |R_h(\xi)| \leqslant \frac{2^{-\frac{3}{2}n}\,\mathrm{e}}{\sqrt{(m+1)}} \{(m+1)\sqrt{n}\}^{m+1} \left(\frac{\sqrt{(8)}\ln f}{m+1}\right)^{(m+1)n} f^{2n+1}.$$

21. These inequalities can be further simplified. First, since $m \geqslant 1$ and $n \geqslant 2$,

$$\frac{2^{-\frac{3}{2}n}\,\mathrm{e}}{\sqrt{(m+1)}} \leqslant \frac{2^{-3}\,\mathrm{e}}{\sqrt{2}} < \frac{1}{2}.$$

Secondly, let $$\kappa(m) = \frac{m}{\sqrt{(m+1)}} \left(\frac{2}{\mathrm{e}}\right)^m.$$

The logarithmic derivative

$$\frac{\mathrm{d} \ln \kappa(m)}{\mathrm{d}m} = \frac{1}{m} - \frac{1}{2(m+1)} + \ln\frac{2}{\mathrm{e}} = \frac{m+2}{2m(m+1)} - 0{\cdot}3068\ldots$$

is positive for $m \leqslant 2$ and negative for $m \geqslant 3$, because it is a decreasing function of m and

$$\left.\frac{\mathrm{d} \ln \kappa(m)}{\mathrm{d}m}\right|_{m=3} < \frac{5}{24} - \frac{3}{10} < 0.$$

Hence, when m runs over the positive integers, $\kappa(m)$ attains its maximum either at $m = 2$ or at $m = 3$, whence

$$\kappa(m) \leqslant \max\left(\frac{8}{\mathrm{e}^2\sqrt{3}}, \frac{12}{\mathrm{e}^3}\right) < 1 \quad \text{for} \quad m = 1, 2, 3, \ldots.$$

It is then clear, by $n \geqslant 2$, that we always have

$$\frac{2^{-\frac{3}{2}n} m}{\sqrt{(m+1)}} \left(\frac{2}{\mathrm{e}}\right)^m < \frac{1}{2}.$$

The inequalities above imply therefore that

$$2 f_1^n m \max_{k=0,1,\ldots,m} (|A_{hk}(\xi)|) < \{(m+1)\,(n+1)^2\}^{m+1} (\sqrt{32})^{(m+1)n} f^n$$

and $$2 f_1^n |R_h(\xi)| < \{(m+1)\sqrt{n}\}^{m+1} \left(\frac{\sqrt{(8)}\ln f}{m+1}\right)^{(m+1)n} f^{2n+1},$$

and so (1) takes the simple form

$$\frac{2}{a} < \{(m+1)\sqrt{n}\}^{m+1} \left(\frac{\sqrt{(8)}\ln f}{m+1}\right)^{(m+1)n} f^{2n+1} + \lambda\{(m+1)\,(n+1)^2\}^{m+1} (\sqrt{32})^{(m+1)n} f^n. \qquad (4)$$

22. The hypothesis (2) implies that

$$m+1>\alpha \ln f, \quad f<e^{\frac{m+1}{\alpha}}, \quad \frac{\sqrt{(8)}\ln f}{m+1}<\frac{\sqrt{8}}{\alpha}.$$

Next, we assumed as part of (3) that $n \geqslant 2$; let us now make the stronger hypothesis that $n \geqslant 6$. Then

$$\{(m+1)\sqrt{n}\}^{m+1} f^{2n+1} < (e^{\frac{\ln(m+1)}{n}+\frac{\ln(n+1)}{2n}+\left(2+\frac{1}{n}\right)\frac{\ln f}{m+1}})^{(m+1)n} < (e^{\frac{2+\frac{1}{6}}{\alpha}+\frac{1}{\beta}+\frac{1}{2\gamma}})^{(m+1)n}$$

and $$\{(m+1)(n+1)^2\}^{m+1} f^n = (e^{\frac{\ln(m+1)}{n}+2\frac{\ln(n+1)}{n}+\frac{\ln f}{m+1}})^{(m+1)n} < (e^{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{2}{\gamma}})^{(m+1)n},$$

and therefore (4) gives the relation

$$\frac{2}{a} < \left(\frac{e^{\frac{13}{6\alpha}+\frac{1}{\beta}+\frac{1}{2\gamma}}\sqrt{8}}{\alpha}\right)^{(m+1)n} + \lambda(e^{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{2}{\gamma}}\sqrt{32})^{(m+1)n}. \tag{5}$$

We shall now try to fix n as a function of a and m such that

$$\left(\frac{e^{\frac{13}{6\alpha}+\frac{1}{\beta}+\frac{1}{2\gamma}}\sqrt{8}}{\alpha}\right)^{(m+1)n} \leqslant \frac{1}{a}, \tag{A}$$

hence, by (5), that also $$\lambda > \{(e^{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{2}{\gamma}}\sqrt{32})^{(m+1)n} a\}^{-1}. \tag{B}$$

If (A) is to hold, we must have

$$\alpha > e^{\frac{13}{6\alpha}+\frac{1}{\beta}+\frac{1}{2\gamma}}\sqrt{8}.$$

It is now easily seen that this inequality can be satisfied by taking

$$\alpha = 10, \quad \beta = 3.$$

For this choice implies that

$$m+1 \geqslant 10 \ln 2 > 6{\cdot}9, \quad \text{hence} \quad m+1 \geqslant 7,$$

and $$n \geqslant 3 \ln 7 > 5{\cdot}8, \quad \text{hence} \quad n \geqslant 6,$$

as required. Also γ may now be chosen as

$$\gamma = \frac{6}{\ln 7},$$

because $\frac{\ln(n+1)}{n}$ is a decreasing function of $n \geqslant 6$, and so

$$n \geqslant \frac{6}{\ln 7}\ln(n+1) \quad \text{if} \quad n \geqslant 6.$$

Therefore $$\frac{e^{\frac{13}{6\alpha}+\frac{1}{\beta}+\frac{1}{2\gamma}}\sqrt{8}}{\alpha} = e^{-0{\cdot}5507\ldots} < e^{-\frac{1}{2}} < 1,$$

as asserted, and also $$e^{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{2}{\gamma}}\sqrt{32} = e^{2{\cdot}8148\ldots} < e^3.$$

We deduce therefore from (A) and (B) that, if

$$e^{\frac{1}{2}(m+1)n} \geqslant a, \tag{a}$$

then $$\lambda > (e^{3(m+1)n} a)^{-1}. \tag{b}$$

23. The inequality (a) is equivalent to

$$n \geqslant \frac{2 \ln a}{m+1}.$$

This condition, and the earlier conditions (3) for n, are all satisfied if n is fixed by the equation

$$n = \max\left([3 \ln (m+1)]+1, \left[\frac{2 \ln a}{m+1}\right]+1\right).$$

Hence the following result has been obtained:

THEOREM 4. *Let f and f_1 be two integers such that $f > f_1 \geqslant 1$ and let*

$$m = [10 \ln f].$$

If a is a positive integer, if $a_1, a_2, \ldots, a_m$ are non-negative integers, and if further

$$n = \max\left([3 \ln (m+1)]+1, \left[\frac{2 \ln a}{m+1}\right]+1\right),$$

then
$$\max_{k=1,2,\ldots,m}\left(\left|\left(\ln \frac{f}{f_1}\right)^k - \frac{a_k}{a}\right|\right) > (e^{3(m+1)n}\,a)^{-1}.$$

It is not difficult to deduce from this theorem a less precise but simpler one involving only the rational approximations to $\ln (f/f_1)$. Denote by a_1/a, where $a \geqslant 1$ and $a_1 \geqslant 0$, a rational approximation to $\ln(f/f_1)$ satisfying the inequality

$$\frac{a_1}{a} \leqslant 2 \ln f;$$

it is obvious that this condition is satisfied as soon as a_1/a is sufficiently near to $\ln (f/f_1.)$ We use the fact that in
$$\left(\ln \frac{f}{f_1}\right)^k - \left(\frac{a_1}{a}\right)^k = \left(\ln \frac{f}{f_1} - \frac{a_1}{a}\right) \sum_{\kappa=0}^{k-1} \left(\ln \frac{f}{f_1}\right)^{k-\kappa-1} \left(\frac{a_1}{a}\right)^{\kappa}$$

the second factor on the right-hand side is in absolute value not greater than

$$\sum_{\kappa=0}^{k-1} (\ln f)^{k-\kappa-1}\,(2 \ln f)^{\kappa} = (\ln f)^{k-1}\,(1+2+2^2+\ldots+2^{k-1}) < 2^k (\ln f)^{k-1}.$$

Hence, for $k = 1, 2, \ldots, m$,

$$\left|\left(\ln \frac{f}{f_1}\right)^k - \left(\frac{a_1}{a}\right)^k\right| \leqslant 2^m\,(\ln f)^{m-1} \left|\ln \frac{f}{f_1} - \frac{a_1}{a}\right|.$$

Apply now the last theorem with the fractions

$$\frac{a_1}{a}, \frac{a_2}{a}, \ldots, \frac{a^m}{a},$$

replaced by
$$\frac{a_1\,a^{m-1}}{a^m}, \quad \frac{a_1^2\,a^{m-2}}{a^m}, \quad \ldots, \quad \frac{a_1^m}{a^m},$$

and the denominator a replaced by the denominator a^m, respectively. It is obvious that the theorem remains true if the value of n is increased. We therefore obtain the following result.

THEOREM 5. *Let f and f_1, a and a_1, be four integers satisfying the inequalities*

$$f > f_1 \geqslant 1, \quad a \geqslant 1, \quad a_1 \geqslant 0, \quad \frac{a_1}{a} \leqslant 2\ln f,$$

and let $$m = [10\ln f], \quad n = \max\left([3\ln(m+1)]+1, [2\ln a]+1\right).$$

Then $$\left|\ln\frac{f}{f_1} - \frac{a_1}{a}\right| > \{2^m(\ln f)^{m-1}\,e^{3(m+1)n}\,a^m\}^{-1}.$$

24. As an application of the last theorem, let us study the expression

$$\Phi = |f^a - f_1^a e^{a_1}|,$$

which may also be written as $$\Phi = f^a\,|1 - e^{-a\lambda}|,$$

where, for shortness, $$\lambda = \ln\frac{f}{f_1} - \frac{a_1}{a}.$$

Suppose, first, $$|\lambda| \geqslant \frac{1}{2a^2}, \quad \text{hence} \quad |a\lambda| \geqslant \frac{1}{2a}.$$

Then $$e^{\frac{1}{2a}} > 1 + \frac{1}{2a} = \frac{2a+1}{2a}, \quad e^{-\frac{1}{2a}} < \frac{2a}{2a+1},$$

and therefore $$|1 - e^{-a\lambda}| \geqslant 1 - e^{-|a\lambda|} \geqslant 1 - e^{-\frac{1}{2a}} > 1 - \frac{2a}{2a+1} = \frac{1}{2a+1},$$

whence $$\Phi \geqslant \frac{f^a}{2a+1} \quad \text{if} \quad |\lambda| \geqslant \frac{1}{2a^2}.$$

Assume, secondly, that

$$|\lambda| < \frac{1}{2a^2}, \quad \text{hence that} \quad |a\lambda| < \frac{1}{2a} \leqslant \frac{1}{2}.$$

It is then not possible that $$\frac{a_1}{a} > 2\ln f,$$
because this inequality implies that

$$|\lambda| = \left|\ln\frac{f}{f_1} - \frac{a_1}{a}\right| > 2\ln f - \ln\frac{f}{f_1} = \ln(ff_1) \geqslant \ln 2 > \frac{1}{2} \geqslant \frac{1}{2a^2}.$$

So, by theorem 5, we have necessarily

$$|\lambda| > \vartheta, \quad \text{where} \quad \vartheta = \{2^m(\ln f)^{m-1}\,e^{3(m+1)n}\,a^m\}^{-1}.$$

Further, by the mean-value theorem of the differential calculus,

$$1 - e^{-a\lambda} = a\lambda\, e^{-a\tau},$$

where τ lies between 0 and λ, so that

$$e^{-a\tau} \geqslant e^{-|a\lambda|} > e^{-\frac{1}{2}} > \tfrac{1}{2}.$$

Therefore, finally, $$\Phi = f^a\,|a\lambda|\,e^{-a\tau} > \tfrac{1}{2}f^a a\vartheta \quad \text{if} \quad |\lambda| < \frac{1}{2a^2}.$$

25. Let, in particular,

$$f = 2, \quad f_1 = 1, \quad \Phi = |2^a - e^{a_1}|, \quad \lambda = \ln 2 - \frac{a_1}{a}.$$

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We shall determine the minimum $M = M(2, e)$ of Φ when a and a_1 run over all pairs of positive integers.

Since $$|2^3 - e^2| = 0{\cdot}6109\ldots, \quad = \alpha \text{ say},$$

this minimum cannot be greater than α. So it only remains to decide whether Φ can assume values less than α for positive integers a, a_1.

If, first, $$|\lambda| \geqslant \frac{1}{2a^2},$$

then $$\Phi \geqslant \frac{2^a}{2a+1},$$

and therefore $$\Phi \geqslant \tfrac{2}{3} > \alpha,$$

as is easily proved by complete induction on a.

Let, secondly, $$|\lambda| < \frac{1}{2a^2},$$

so that $$\Phi > 2^{a-1} a\vartheta.$$

Here, by theorem 5,

$$m = [10 \ln 2] = 6, \quad [3 \ln (m+1)] + 1 = 6, \quad n = \max(6, [2 \ln a] + 1),$$

and therefore

$$n = [2 \ln a] + 1 \leqslant 2 \ln a + 1 \quad \text{if} \quad a \geqslant 13, \quad \text{i.e. if} \quad [2 \ln a] \geqslant 5.$$

Further $$\vartheta \geqslant \{2^6 (\ln 2)^5 \, e^{3 \times 7 (2 \ln a + 1)} a^6\}^{-1} = \{2^6 (\ln 2)^5 \, e^{21} a^{48}\}^{-1}.$$

Since $$2^6 (\ln 2)^5 < e^3,$$

Φ satisfies then the inequality

$$\Phi > 2^{a-1} e^{-24} a^{-47}, \quad = \phi(a) \text{ say}, \quad \text{if} \quad a \geqslant 13.$$

Here $\phi(a)$ is an increasing function of a if

$$\frac{d \ln \phi(a)}{da} = \ln 2 - \frac{47}{a} \geqslant 0,$$

thus certainly if $a \geqslant 512$. Since

$$\phi(512) = 2^{511} e^{-24} 512^{-47} = 2^{88} e^{-24} > 1 > \alpha,$$

we find therefore that $$\Phi > \alpha \quad \text{if} \quad a \geqslant 512.$$

It follows that any possible solution of

$$\Phi \leqslant \alpha$$

belongs to a value of a less than 512; moreover,

$$|\lambda| = \left| \ln 2 - \frac{a_1}{a} \right| < \frac{1}{2a^2}.$$

Therefore, by the theory of continued fractions, a_1/a is one of the finite set of convergents

$$\frac{a_1}{a} = \frac{1}{1}, \quad \frac{2}{3}, \quad \frac{7}{10}, \quad \frac{9}{13}, \quad \frac{61}{88}, \quad \frac{192}{277}, \quad \frac{253}{365}$$

of the continued fraction

$$\ln 2 = \frac{1|}{|1} + \frac{1|}{|2} + \frac{1|}{|3} + \frac{1|}{|1} + \frac{1|}{|6} + \frac{1|}{|3} + \frac{1|}{|1} + \dots$$

for $\ln 2$. The table

$$\begin{aligned} |2^1 - e^1| &= 0{\cdot}718\ldots > \alpha, \\ |2^3 - e^2| &= \alpha, \\ |2^{10} - e^7| &> 72 > \alpha, \\ |2^{13} - e^9| &> 88 > \alpha, \\ |2^{88} - e^{61}| &> 9\times 10^{23} > \alpha, \\ |2^{277} - e^{192}| &> 8\times 10^{86} > \alpha, \\ |2^{365} - e^{253}| &> 9\times 10^{112} > \alpha, \end{aligned}$$

shows then that the minimum of Φ is attained for $a = 3$, $a_1 = 2$ and that

$$|2^a - e^{a_1}| \geqslant |2^3 - e^2|$$

for all pairs of positive integers a, a_1, with equality only in this obvious case.

I am greatly indebted to Mr D. F. Ferguson, M.A., for determining by the same method the following extreme values:

$$\begin{aligned} |3^a - e^{a_1}| &\geqslant |3^1 - e^1| = 0{\cdot}281\ldots, \\ |4^a - e^{a_1}| &\geqslant |4^1 - e^1| = 1{\cdot}281\ldots, \\ |5^a - e^{a_1}| &\geqslant |5^1 - e^1| = 2{\cdot}281\ldots, \\ |6^a - e^{a_1}| &\geqslant |6^1 - e^2| = 1{\cdot}389\ldots, \\ |7^a - e^{a_1}| &\geqslant |7^1 - e^2| = 0{\cdot}389\ldots, \\ |20^a - e^{a_1}| &\geqslant |20^1 - e^3| = 0{\cdot}085\ldots, \\ |90^a - e^{a_1}| &\geqslant |90^2 - e^9| = 3{\cdot}083\ldots. \end{aligned}$$

Chapter 4. The logarithms of integers

26. Let f be a very large positive integer and a_1 an arbitrary integer. On putting $f_1 = 1$ and $a = 1$ in theorem 5, the following result is obtained:*

If $\qquad m = [10 \ln f] \quad \text{and} \quad n = [3 \ln (m+1)] + 1,$

then $\qquad |\ln f - a_1| > \{2^m (\ln f)^{m-1}\, e^{3(m+1)n}\}^{-1}.$

In this inequality, $\qquad 2^m (\ln f)^{m-1} < (2 \ln f)^m \leqslant e^{10 \ln f (\ln \ln f + \ln 2)}$

and $\qquad e^{3(m+1)n} \leqslant e^{3(10 \ln f + 1)\{3 \ln (10 \ln f + 1) + 1\}}.$

Therefore, for sufficiently large f,

$$|\ln f - a_1| > f^{-c \ln \ln f},$$

where c may be any constant greater than $10 + 90 = 100$. In the present chapter, we shall improve on this estimate by a slight change in the computations of the last chapter.

* The condition $a_1 \leqslant 2 \ln f$ of the theorem may evidently be disregarded.

27. For this purpose, denote by f a very large positive integer, by $\alpha \geqslant 2$ and β two positive constants to be selected at once, by m and n the integers

$$m = [\alpha \ln f], \quad n = [\beta \ln (m+1)]+1,$$

and by $a_1, a_2, \ldots, a_m$ arbitrary integers; further put

$$\lambda = \max_{k=1,2,\ldots,m} (|(\ln f)^k - a_k|).$$

The definitions of m and n imply that

$$n \geqslant \gamma \ln (n+1), \quad n \geqslant \frac{1}{\delta}$$

for any two given positive constants γ and δ, provided only f is already large enough.

Therefore a trivial change in the estimations in §21 and §22 leads immediately to the inequality

$$2 < \left(\frac{e^{\frac{2+\delta}{\alpha}+\frac{1}{\beta}+\frac{1}{2\gamma}}\sqrt{8}}{\alpha}\right)^{(m+1)n} + \lambda\, (e^{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{2}{\gamma}}\sqrt{32})^{(m+1)n}.$$

It is then clear that

$$\lambda > (e^{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{2}{\gamma}}\sqrt{32})^{-(m+1)n},$$

provided that

$$\alpha \geqslant e^{\frac{2+\delta}{\alpha}+\frac{1}{\beta}+\frac{1}{2\gamma}}\sqrt{8}.$$

Choose now

$$\alpha = 10, \quad \beta = 1.$$

A trivial calculation shows that

$$\alpha > e^{\frac{2}{\alpha}+\frac{1}{\beta}}\sqrt{8}, \quad (e^{\frac{1}{\alpha}+\frac{1}{\beta}}\sqrt{32})^{\alpha\beta} < e^{29},$$

and so also

$$\alpha \geqslant e^{\frac{2+\delta}{\alpha}+\frac{1}{\beta}+\frac{1}{2\gamma}}\sqrt{8}, \quad (e^{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{2}{\gamma}}\sqrt{32})^{\alpha\beta} < e^{29},$$

if only $\gamma > 0$ is sufficiently large and $\delta > 0$ is sufficiently small, as shall be assumed from now on. Since

$$m \sim \alpha \ln f, \quad n \sim \beta \ln \ln f,$$

as f tends to infinity, we have thus the following result:

THEOREM 6. *Let f be a sufficiently large positive integer. If $m = [10 \ln f]$, and if $a_1, a_2, \ldots, a_m$ are m arbitrary integers, then*

$$\max_{k=1,2,\ldots,m} (|(\ln f)^k - a_k|) > f^{-29 \ln \ln f}.$$

28. With a slight change of notation, denote now by a the integer nearest to $\ln f$, and put

$$\lambda = \ln f - a,$$

so that

$$-\tfrac{1}{2} \leqslant \lambda \leqslant +\tfrac{1}{2}$$

and therefore

$$\max (\ln f, a) \leqslant \ln f + \tfrac{1}{2} = \left\{1 + \frac{1}{2 \ln f}\right\} \ln f.$$

Then

$$\left|\frac{(\ln f)^k - a^k}{\ln f - a}\right| = (\ln f)^{k-1} + a(\ln f)^{k-2} + \ldots + a^{k-1} \leqslant k\{\max (\ln f, a)\}^{k-1},$$

whence, for $k = 1, 2, \ldots, m$,

$$\left|\frac{(\ln f)^k - a^k}{\ln f - a}\right| \leqslant m\{\max (\ln f, a)\}^m \leqslant m\left\{1 + \frac{1}{2 \ln f}\right\}^m (\ln f)^m.$$

Since $m = [10\ln f]$, $\quad \left\{1+\frac{1}{2\ln f}\right\}^m < \left\{\exp\left(\frac{1}{2\ln f}\right)\right\}^{10\ln f} = e^5.$

Hence $$m\left\{1+\frac{1}{2\ln f}\right\}^m (\ln f)^m < 10\ln f \,.\, e^5 (\ln f)^{10\ln f} < (\ln f)^{11\ln f}$$

as soon as f is sufficiently large. We find then that

$$\left|\frac{(\ln f)^k - a^k}{\ln f - a}\right| < (\ln f)^{11\ln f} = f^{11\ln\ln f} \quad (k = 1, 2, \ldots, m),$$

and therefore $$|\ln f - a| > f^{-11\ln\ln f} \max_{k=1,2,\ldots,m} (|(\ln f)^k - a^k|).$$

On the other hand, $$\max_{k=1,2,\ldots,m} (|(\ln f)^k - a^k|) > f^{-29\ln\ln f}$$

by theorem 6 applied with $a_k = a^k$ for $k = 1, 2, \ldots, m$. We combine these two inequalities and note that the resulting formula remains true even when the integer a is not the one nearest to $\ln f$. Hence we find:

THEOREM 7. *If f is a sufficiently large positive integer and if a is an arbitrary integer, then*

$$|\ln f - a| > f^{-40\ln\ln f}.$$

The exponent $40\ln\ln f$ tends to infinity very slowly; the theorem is thus not excessively weak, the more so since one can easily show that

$$|\ln f - a| < \frac{1}{f}$$

for an infinite increasing sequence of positive integers f and suitable integers a.

29. By means of the last result it is possible to determine a lower bound for the fractional parts of the powers of e.

Denote by a a large positive integer and by f the integer nearest to e^a; therefore

$$e^a - \tfrac{1}{2} \leqslant f \leqslant e^a + \tfrac{1}{2}.$$

By the mean value theorem of differential calculus,

$$\frac{e^a - f}{a - \ln f} = e^\alpha,$$

where α is a certain number between a and $\ln f$, hence e^α a number between e^a and f. Therefore

$$e^\alpha \geqslant e^a - \tfrac{1}{2} > \tfrac{1}{2}e^a,$$

whence $$|e^a - f| > \tfrac{1}{2}e^a |a - \ln f|,$$

and so theorem 7 implies that

$$|e^a - f| > \tfrac{1}{2}e^a f^{-40\ln\ln f} = \tfrac{1}{2}e^a e^{-40\ln f \ln\ln f}.$$

Here $$f \leqslant e^a + \tfrac{1}{2}, \quad \ln f \leqslant a + \ln(1 + \tfrac{1}{2}e^{-a}) \leqslant a + \tfrac{1}{2}e^{-a},$$

and $$\ln\ln f \leqslant \ln a + \ln\left(1 + \frac{1}{2a}e^{-a}\right) \leqslant \ln a + \frac{1}{2a}e^{-a}.$$

Hence $$\ln f \ln\ln f \leqslant (a+\tfrac{1}{2}e^{-a})\left(\ln a+\frac{1}{2a}e^{-a}\right) = a\ln a+\tfrac{1}{2}\ln(ea)\,e^{-a}+\frac{1}{4a}e^{-2a},$$

and finally $$\tfrac{1}{2}e^{a}f^{-40\ln\ln f} > e^{-40a\ln a}$$

as soon as a and therefore f are sufficiently large. Similarly as in the last section we may drop the condition that f is the integer nearest to e^a. The result is therefore as follows.

THEOREM 8. *If a is a sufficiently large positive integer and if f is an arbitrary integer, then*

$$|e^a - f| > a^{-40a}.$$

This estimate is rather weak, but it does not seem easy to obtain any substantial improvement.

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